السؤال الأول

(1) $$\int \frac{2x}{(x^2+3)(x^2+1)}\,dx$$

الحل

$$x^2 = u \longrightarrow 2x\,dx = du$$

$$\int \frac{du}{(u+3)(u+1)} =$$

$$\frac{1}{(u+3)(u+1)} = \frac{A}{u+3} + \frac{B}{u+1}$$

$$1 = A(u+1) + B(u+3)$$

نضع $u=-1$

$$1 = 2B \Rightarrow B = \frac{1}{2}$$

نضع $u=-3$

$$1 = -2A \Rightarrow A = -\frac{1}{2}$$

$$\therefore \int \frac{du}{(u+3)(u+1)} = -\frac{1}{2}\int \frac{du}{u+3} + \frac{1}{2}\int \frac{1}{u+1}\,du$$

$$= -\frac{1}{2}\ln|u+3| + \frac{1}{2}\ln|u+1| + c$$

$$= \frac{1}{2}\ln\left|\frac{u+1}{u+3}\right| + c$$

$$\therefore \int \frac{2x}{(x^2+3)(x^2+1)}\,dx = \frac{1}{2}\ln\left|\frac{x^2+1}{x^2+3}\right| + c$$

(2) أوجد $\int \frac{1}{x(x^4-1)}\,dx$

أولا : نضرب البسط والمقام في $x$

$$\int \frac{x}{x^2(x^4-1)}\,dx$$

نضع

$$u = x^2 \Rightarrow du = 2x\,dx \Rightarrow x\,dx = \frac{du}{2}$$

$$\therefore \int \frac{du}{2u(u^2-1)} = \frac{1}{2}\int \frac{du}{u(u-1)(u+1)}$$

$$\frac{1}{u(u-1)(u+1)} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{u+1}$$

$$1 = A(u-1)(u+1) + Bu(u+1) + Cu(u-1)$$

نضع $u=0$

$$1 = A$$

نضع $u=1$

$$1 = 2B \Rightarrow B = \frac{1}{2}$$

نضع $u=-1$

$$1 = 2C \qquad C = \frac{1}{2}$$

$$\therefore \frac{1}{2}\int \frac{du}{u(u-1)(u+1)} = \frac{1}{2}\left[\int \frac{1}{u}\,du + \frac{1}{2}\int\frac{du}{u-1} + \frac{1}{2}\int\frac{du}{u+1}\right]$$

$$= \frac{1}{2}\left[\ln|u| + \frac{1}{2}\ln|u-1| + \frac{1}{2}\ln|u+1|\right] + c$$

$$= \frac{1}{2}\left[\ln\left|u\sqrt{u^2-1}\right|\right] + c$$

$$\therefore \int \frac{x}{x^2(x^4-1)}\,dx = \frac{1}{2}\ln\left|x^2\sqrt{x^4-1}\right| + c$$

ثانيا بضرب البسط والمقام في $x^3$

$$\int \frac{x^3}{x^4(x^4-1)}\,dx$$

بوضع

$$u = x^4 \Rightarrow du = 4x^3dx \Rightarrow x^3dx = \frac{du}{4}$$

$$\int \frac{x^3}{x^4(x^4-1)}\,dx = \frac{1}{4}\int \frac{du}{u(u-1)}$$

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$

$$1 = A(u-1) + Bu$$

بوضع $u=0$

$$1 = -A \Rightarrow A = -1$$

بوضع $u=1$

$$1 = B$$

$$\therefore \frac{1}{4}\int \frac{du}{u(u-1)} = \frac{1}{4}\left[\int \frac{-1}{u}du + \int \frac{1}{u-1}du\right]$$

$$= \frac{1}{4}\left[-\ln|u| + \ln|u-1|\right] + C$$

$$= \frac{1}{4}\ln\left|\frac{u-1}{u}\right| + C$$

$$\therefore \int \frac{x^3}{x^4(x^4-1)}\,dx = \frac{1}{4}\ln\left|\frac{x^4-1}{x^4}\right| + C$$

③ أوجد $\int \frac{1}{x(x^n+1)}\,dx$

حيث $n$ عدد صحيح موجب

بضرب البسط والمقام $x^{n-1}$

$$\int \frac{x^{n-1}\,dx}{x^n(x^n+1)}$$

بوضع

$$u = x^n \Rightarrow du = n x^{n-1}dx \Rightarrow x^{n-1}dx = \frac{du}{n}$$

$$\therefore \int \frac{x^{n-1}dx}{x^n(x^n+1)} = \frac{1}{n}\int \frac{du}{u(u+1)}$$

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1} \Rightarrow 1 = A(u+1) + Bu$$

بوضع $u=0$

$$1 = A$$

بوضع $u=-1$

$$\therefore B = -1$$

$$\therefore \frac{1}{n}\int \frac{du}{u(u+1)} = \frac{1}{n}\left[\int \frac{1}{u}du - \int \frac{1}{u+1}du\right]$$

$$= \frac{1}{n}\left[\ln|u| - \ln|u+1|\right] + C$$

$$= \frac{1}{n}\ln\left|\frac{u}{u+1}\right| + C$$

$$\therefore \int \frac{1}{x(x^n+1)}dx = \frac{1}{n}\ln\left|\frac{x^n}{x^n+1}\right| + C$$

السؤال الثاني

(١) أوجد $\int \frac{1}{x^3(x^2+1)}\,dx$

بضرب البسط والمقام في x

$$\int \frac{x}{x^4(x^2+1)}\,dx =$$

بوضع $u = x^2$

$$du = 2x\,dx \Rightarrow x\,dx = \frac{du}{2}$$

$$\therefore \int \frac{x\,dx}{x^4(x^2+1)} = \frac{1}{2}\int \frac{du}{u^2(u+1)}$$

$$\frac{1}{u^2(u+1)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u+1}$$

$$1 = Au(u+1) + B(u+1) + Cu^2$$

بوضع $u = 0$

$$B = 1$$

بوضع $u = -1$

$$C = 1$$

بوضع $u = 2$

$$\therefore 1 = 6A + 3B + 4C \Rightarrow A = -1$$

$$\therefore \frac{1}{2}\int \frac{du}{u^2(u+1)} = \frac{1}{2}\left[\int \frac{-1}{u}\,du + \int \frac{1}{u^2}\,du + \int \frac{1}{u+1}\,du\right]$$

$$= \frac{1}{2}\left[-\ln|u| - \frac{1}{u} + \ln|u+1| + c\right.$$

$$= \frac{1}{2}\left[\ln\left|\frac{u+1}{u}\right| + -\frac{1}{u}\right] + C$$

$$\therefore \int \frac{1}{x^3(x^2+1)}\,dx = \frac{1}{2}\left[\ln\left|\frac{x^2+1}{x^2}\right| - \frac{1}{x^2}\right] + C$$

(2) أوجد

$$I=\int(\cot x)^3\,dx$$

باستخدام التعويض $u=\tan x$

الحل

$$I=\int\frac{1}{(\tan x)^3}\,dx$$

$$du=\sec^2 x\,dx$$

$$du=(\tan^2 x+1)\,dx$$

$$du=(u^2+1)\,dx \Rightarrow dx=\frac{du}{u^2+1}$$

$$I=\int\frac{du}{u^3(u^2+1)}$$

من المسألة السابقة

$$I=\frac{1}{2}\left[\ln\left|\frac{u^2+1}{u^2}\right|-\frac{1}{u^2}\right]+C$$

$$I=\frac{1}{2}\left[\ln\left|\frac{\tan^2 x+1}{\tan^2 x}\right|-\frac{1}{\tan^2 x}\right]+C$$

$$=\frac{1}{2}\left[\ln|\csc^2 x|-\cot^2 x\right]+C$$

(3) أوجد

$$\int \frac{\cos x}{(1-\sin x)(2-\sin x)}\,dx$$

الحل

بوضع $u = \sin x$

$du = \cos x\,dx \Longrightarrow \cos x\,dx = du$

$$\therefore \int \frac{du}{(1-u)(2-u)}$$

$$\frac{1}{(1-u)(2-u)} = \frac{1}{(u-1)(u-2)} = \frac{A}{u-1} + \frac{B}{u-2}$$

$$1 = A(u-2) + B(u-1)$$

بوضع $u=2$

$1 = B$

بوضع $u=1$

$A = -1$

$$\int \frac{du}{(u-1)(u-2)} = -\int \frac{1}{u-1}\,du + \int \frac{1}{u-2}\,du$$

$$= -\ln|u-1| + \ln|u-2| + C$$

$$= \ln\left|\frac{u-2}{u-1}\right| + C$$

$$\therefore \int \frac{\cos x}{(1-\sin x)(2-\sin x)}\,dx = \ln\left|\frac{\sin x-2}{\sin x-1}\right| + C$$

السؤال الثالث

(1) أوجد

$$\int \frac{x^2}{1-x^6}\,dx$$

الحل

نفرض أن

$$u = x^3 \Rightarrow du = 3x^2\,dx \Rightarrow x^2\,dx = \frac{du}{3}$$

$$\therefore \int \frac{x^2}{1-x^6}\,dx = \frac{1}{3}\int \frac{du}{1-u^2} = \frac{1}{3}\int \frac{du}{(1-u)(1+u)}$$

$$\frac{1}{(1-u)(1+u)} = \frac{A}{1-u} + \frac{B}{1+u}$$

$$1 = A(1+u) + B(1-u)$$

بوضع $u = -1$

$$1 = 2B \Rightarrow B = \frac{1}{2}$$

بوضع $u = 1$

$$A = \frac{1}{2}$$

$$\therefore \frac{1}{3}\int \frac{du}{(1-u)(1+u)} = \frac{1}{3}\left[\frac{1}{2}\int \frac{1}{1-u}\,du + \frac{1}{2}\int \frac{1}{1+u}\,du\right]$$

$$= \frac{1}{3}\left[-\frac{1}{2}\ln|1-u| + \frac{1}{2}\ln|1+u|\right] + c$$

$$= \frac{1}{3}\ln\left|\sqrt{\frac{1+u}{1-u}}\right| + c$$

$$\therefore \int \frac{x^2}{1-x^6}\,dx = \frac{1}{3}\ln\left|\sqrt{\frac{1+x^3}{1-x^3}}\right| + c$$

(2)

$$\int \frac{x^{n-1}}{1-x^{2n}}\,dx$$

الحل

بوضع

$$u = x^{n} \Longrightarrow du = n\,x^{n-1}\,dx$$

$$\therefore\; x^{n-1}dx = \frac{du}{n}$$

من المسألة السابقة

$$\therefore \frac{1}{n}\int \frac{du}{1-u^{2}} = \frac{1}{n}\ln\left|\sqrt{\frac{1+u}{1-u}}\right| + c$$

$$= \frac{1}{n}\ln\left|\sqrt{\frac{1+x^{n}}{1-x^{n}}}\right| + c$$

③ أوجد (بأكثر من ثلاث طرق)

$$\int \frac{1}{e^{x}-1}\,dx$$

الطريقه الأولى بضرب البسط والمقام في $e^{-x}$

$$\therefore \int \frac{e^{-x}}{1-e^{-x}}\,dx = \ln|1-e^{-x}| + c$$

الطريقة الثانيه

$$\int \frac{1}{e^{x}-1}\,dx = \int \frac{e^{x}+1-e^{x}}{e^{x}-1}\,dx =$$

$$= \int \frac{e^{x}dx}{e^{x}-1} - \int 1\,dx = \ln|e^{x}-1| - x + c$$

الطريقة الثالثة

$$\int \frac{1}{e^x-1}\,dx$$

الحل

بوضع $u = e^x$

$du = e^x dx \Rightarrow du = u\,dx$

$\therefore dx = \frac{du}{u}$

$$\therefore \int \frac{du}{u(u-1)} = \int \frac{A}{u}du + \int \frac{B}{u-1}du$$

$1 = A(u-1) + Bu$

بوضع $u=1$

$\therefore B = 1$

بوضع $u=0$

$\therefore A = -1$

$$\therefore \int \frac{du}{u(u-1)} = -\ln|u| + \ln|u-1| + c$$

$$= \ln\left|\frac{u-1}{u}\right| + c$$

$$\therefore \int \frac{1}{e^x-1}\,dx = \ln\left|\frac{e^x-1}{e^x}\right| + c$$