# 2.4 Analyzing Graphs of Quadratic Functions

- *Find the vertex, the axis of symmetry, and the maximum or minimum value of a quadratic function using the method of completing the square.*
- *Graph quadratic functions.*
- *Solve applied problems involving maximum and minimum function values.*

## Graphing Quadratic Functions of the Type $f(x) = a(x - h)^2 + k$

The graph of a quadratic function is called a **parabola**. The graph of every parabola evolves from the graph of the squaring function $f(x) = x^2$ using transformations.

**EXPLORING WITH TECHNOLOGY** Think of transformations and look for patterns. Consider the following functions:

$$y_1 = x^2, \qquad y_2 = -0.4x^2,$$
$$y_3 = -0.4(x-2)^2, \qquad y_4 = -0.4(x-2)^2 + 3.$$

Graph $y_1$ and $y_2$. How do you get from the graph of $y_1$ to $y_2$?
Graph $y_2$ and $y_3$. How do you get from the graph of $y_2$ to $y_3$?
Graph $y_3$ and $y_4$. How do you get from the graph of $y_3$ to $y_4$?

Consider the following functions:

$$y_1 = x^2, \qquad y_2 = 2x^2,$$
$$y_3 = 2(x+3)^2, \qquad y_4 = 2(x+3)^2 - 5.$$

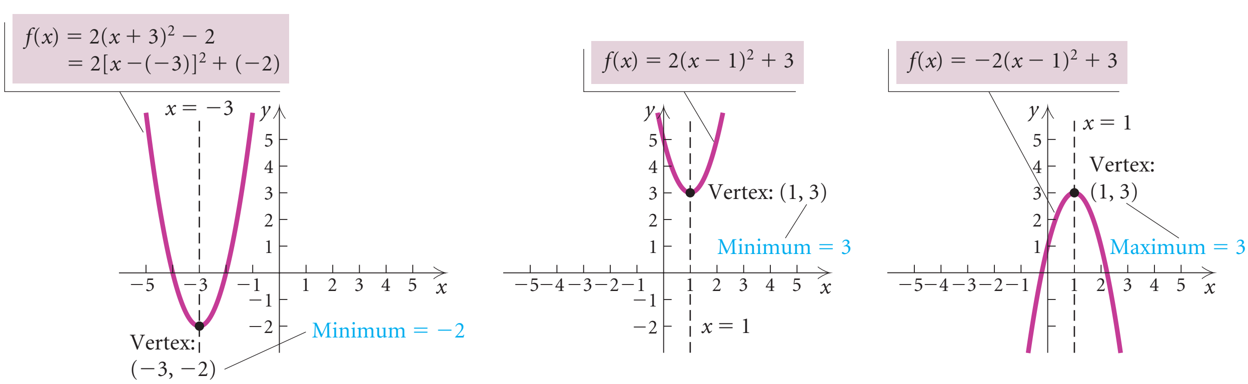
Graph $y_1$ and $y_2$. How do you get from the graph of $y_1$ to $y_2$?
Graph $y_2$ and $y_3$. How do you get from the graph of $y_2$ to $y_3$?
Graph $y_3$ and $y_4$. How do you get from the graph of $y_3$ to $y_4$?

**TRANSFORMATIONS**
REVIEW SECTION 1.7.

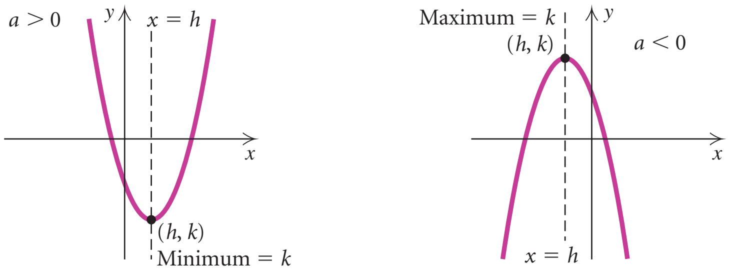
We get the graph of $f(x) = a(x-h)^2 + k$ from the graph of $f(x) = x^2$ as follows:

$$f(x) = x^2$$
$$\downarrow$$
$$f(x) = ax^2$$ **Vertical stretching or shrinking with a reflection across the $x$-axis if $a < 0$**
$$\downarrow$$
$$f(x) = a(x-h)^2$$ **Horizontal translation**
$$\downarrow$$
$$f(x) = a(x-h)^2 + k.$$ **Vertical translation**

Consider the following graphs of the form $f(x) = a(x-h)^2 + k$. The point $(h, k)$ at which the graph turns is called the **vertex**. The maximum or minimum value of $f(x)$ occurs at the vertex. Each graph has a line $x = h$ that is called the **axis of symmetry**.



## CONNECTING THE CONCEPTS




### GRAPHING QUADRATIC FUNCTIONS

The graph of the function $f(x) = a(x - h)^2 + k$ is a parabola that

- opens up if $a > 0$ and down if $a < 0$;
- has $(h, k)$ as the vertex;
- has $x = h$ as the axis of symmetry;
- has $k$ as a minimum value (output) if $a > 0$;
- has $k$ as a maximum value if $a < 0$.

As we saw in Section 1.7, the constant $a$ serves to stretch or shrink the graph vertically. As a parabola is stretched vertically, it becomes narrower, and as it is shrunk vertically, it becomes wider. That is, as $|a|$ increases, the graph becomes narrower, and as $|a|$ gets close to 0, the graph becomes wider.

If the equation is in the form $f(x) = a(x - h)^2 + k$, we can learn a great deal about the graph without graphing.

| FUNCTION | $f(x) = 3\left(x - \frac{1}{4}\right)^2 - 2$ $= 3\left(x - \frac{1}{4}\right)^2 + (-2)$ | $g(x) = -3(x + 5)^2 + 7$ $= -3[x - (-5)]^2 + 7$ |
|---|---|---|
| VERTEX | $\left(\frac{1}{4}, -2\right)$ | $(-5, 7)$ |
| AXIS OF SYMMETRY | $x = \frac{1}{4}$ | $x = -5$ |
| MAXIMUM | None ($3 > 0$, so graph opens up.) | 7 ($-3 < 0$, so graph opens down.) |
| MINIMUM | $-2$ ($3 > 0$, so graph opens up.) | None ($-3 < 0$, so graph opens down.) |

Note that the vertex $(h, k)$ is used to find the maximum or the minimum value of the function. The maximum or minimum value is the number $k$, *not* the ordered pair $(h, k)$.

## Graphing Quadratic Functions of the Type $f(x) = ax^2 + bx + c, a \neq 0$

We now use a modification of the method of completing the square as an aid in graphing and analyzing quadratic functions of the form $f(x) = ax^2 + bx + c$, $a \neq 0$.

Animation Video You Try It

**EXAMPLE 1** Find the vertex, the axis of symmetry, and the maximum or minimum value of $f(x) = x^2 + 10x + 23$. Then graph the function.

*Solution* To express $f(x) = x^2 + 10x + 23$ in the form $f(x) = a(x - h)^2 + k$, we complete the square on the terms involving $x$. To do so, we take half the coefficient of $x$ and square it, obtaining $(10/2)^2$, or 25. We now add and subtract that number on the *right side*:

$$f(x) = x^2 + 10x + 23 = x^2 + 10x + 25 - 25 + 23.$$

Since $25 - 25 = 0$, the new expression for the function is equivalent to the original expression. Note that this process differs from the one we used to complete the square in order to solve a quadratic equation, where we added the same number on both sides of the equation to obtain an equivalent equation. Instead, when we complete the square to write a function in the form $f(x) = a(x - h)^2 + k$, we add and subtract the same number on the right side. The entire process is shown below:

$$f(x) = x^2 + 10x + 23$$ **Note that 25 completes the square for $x^2 + 10x$.**

$$= x^2 + 10x + 25 - 25 + 23$$ **Adding $25 - 25$, or 0, to the right side**

$$= (x^2 + 10x + 25) - 25 + 23$$ **Regrouping**

$$= (x + 5)^2 - 2$$ **Factoring and simplifying**

$$= [x - (-5)]^2 + (-2).$$ **Writing in the form $f(x) = a(x - h)^2 + k$**

Keeping in mind that this function will have a minimum value since $a > 0$ $(a = 1)$, from this form of the function we know the following:

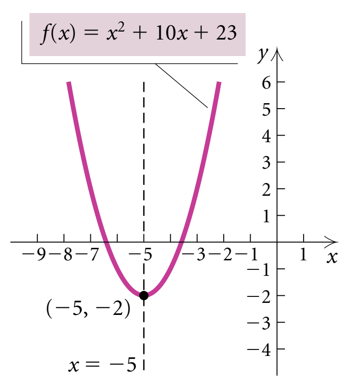
Vertex: $(-5, -2)$;

Axis of symmetry: $x = -5$;

Minimum value of the function: $-2$.

To graph the function by hand, we first plot the vertex and find several points on either side of it. Then we plot these points and connect them with a smooth curve.

| $x$ | $f(x)$ | |
|---|---|---|
| $-5$ | $-2$ | ← Vertex |
| $-4$ | $-1$ | |
| $-2$ | 7 | |
| $-7$ | 2 | |
| $-8$ | 7 | |




The graph of $f(x) = x^2 + 10x + 23$, or $[x - (-5)]^2 + (-2)$, shown above, is a shift of the graph of $y = x^2$ left 5 units and down 2 units.

Keep in mind that the axis of symmetry is not part of the graph; it is a characteristic of the graph. If you fold the graph on its axis of symmetry, the two halves of the graph will coincide.

**EXAMPLE 2** Find the vertex, the axis of symmetry, and the maximum or minimum value of $g(x) = x^2/2 - 4x + 8$. Then graph the function.

*Solution* We complete the square in order to write the function in the form $g(x) = a(x - h)^2 + k$. First, we factor $\frac{1}{2}$ out of the first two terms. This makes the coefficient of $x^2$ within the parentheses 1:

$$g(x) = \frac{x^2}{2} - 4x + 8$$
$$= \frac{1}{2}(x^2 - 8x) + 8.$$

**Factoring $\frac{1}{2}$ out of the first two terms: $\frac{x^2}{2} - 4x = \frac{1}{2} \cdot x^2 - \frac{1}{2} \cdot 8x$**

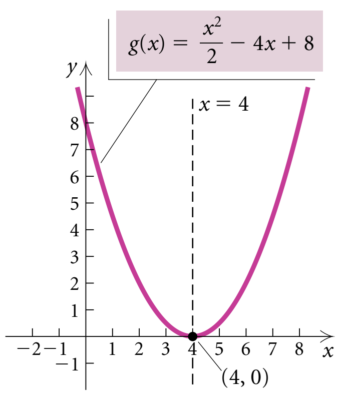
Next, we complete the square inside the parentheses: Half of $-8$ is $-4$, and $(-4)^2 = 16$. We add and subtract 16 inside the parentheses:

$$g(x) = \tfrac{1}{2}(x^2 - 8x + 16 - 16) + 8$$
$$= \tfrac{1}{2}(x^2 - 8x + 16) - \tfrac{1}{2} \cdot 16 + 8$$

**Using the distributive law to remove $-16$ from within the parentheses**

$$= \tfrac{1}{2}(x - 4)^2 + 0, \text{ or } \tfrac{1}{2}(x - 4)^2.$$

**Factoring and simplifying**




We know the following:

Vertex: $(4, 0)$;
Axis of symmetry: $x = 4$;
Minimum value of the function: 0.

Finally, we plot the vertex and several points on either side of it and draw the graph of the function. The graph of $g$ is a vertical shrinking of the graph of $y = x^2$ along with a shift 4 units to the right.

**EXAMPLE 3** Find the vertex, the axis of symmetry, and the maximum or minimum value of $f(x) = -2x^2 + 10x - \frac{23}{2}$. Then graph the function.

*Solution* We have

$$f(x) = -2x^2 + 10x - \tfrac{23}{2}$$
$$= -2(x^2 - 5x) - \tfrac{23}{2}$$

**Factoring $-2$ out of the first two terms**

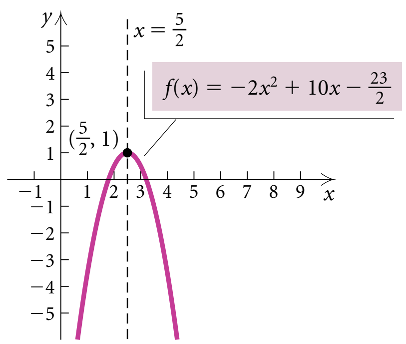
$$= -2\left(x^2 - 5x + \tfrac{25}{4} - \tfrac{25}{4}\right) - \tfrac{23}{2}$$

**Completing the square inside the parentheses**

$$= -2\left(x^2 - 5x + \tfrac{25}{4}\right) - 2\left(-\tfrac{25}{4}\right) - \tfrac{23}{2}$$

**Using the distributive law to remove $-\frac{25}{4}$ from within the parentheses**

$$= -2\left(x - \tfrac{5}{2}\right)^2 + \tfrac{25}{2} - \tfrac{23}{2}$$
$$= -2\left(x - \tfrac{5}{2}\right)^2 + 1.$$

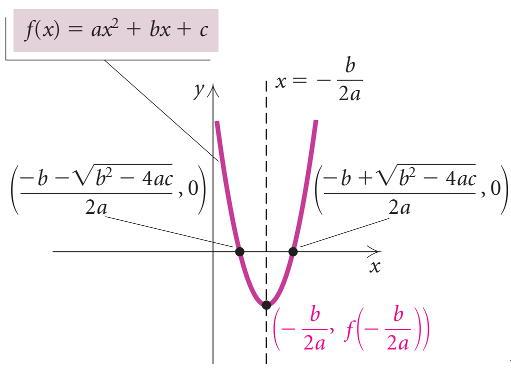



This form of the function yields the following:

Vertex: $\left(\frac{5}{2}, 1\right)$;
Axis of symmetry: $x = \frac{5}{2}$;
Maximum value of the function: 1.

The graph is found by shifting the graph of $f(x) = x^2$ to the right $\frac{5}{2}$ units, reflecting it across the $x$-axis, stretching it vertically, and shifting it up 1 unit.

In many situations, we want to find the coordinates of the vertex directly from the equation $f(x) = ax^2 + bx + c$ using a formula. One way to develop such a formula is to observe that the $x$-coordinate of the vertex is centered between the $x$-intercepts, or zeros, of the function. By averaging the two solutions of $ax^2 + bx + c = 0$, we find a formula for the $x$-coordinate of the vertex:




$$x\text{-coordinate of vertex} = \frac{\dfrac{-b - \sqrt{b^2 - 4ac}}{2a} + \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}}{2}$$
$$= \frac{\dfrac{-2b}{2a}}{2} = \frac{-\dfrac{b}{a}}{2}$$
$$= -\frac{b}{a} \cdot \frac{1}{2} = -\frac{b}{2a}.$$

We use this value of $x$ to find the $y$-coordinate of the vertex, $f\left(-\dfrac{b}{2a}\right)$.

### *The Vertex of a Parabola*

The **vertex** of the graph of $f(x) = ax^2 + bx + c$ is

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right).$$

We calculate the $x$-coordinate. We substitute to find the $y$-coordinate.

Video

**EXAMPLE 4** For the function $f(x) = -x^2 + 14x - 47$:

**a)** Find the vertex.

**b)** Determine whether there is a maximum or minimum value and find that value.

**c)** Find the range.

**d)** On what intervals is the function increasing? decreasing?

*Solution* There is no need to graph the function.

**a)** The $x$-coordinate of the vertex is

$$-\frac{b}{2a} = -\frac{14}{2(-1)}, \text{ or } 7.$$

Since

$$f(7) = -7^2 + 14 \cdot 7 - 47 = 2,$$

the vertex is $(7, 2)$.

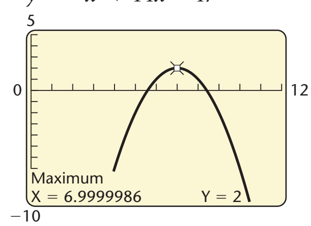
**b)** Since $a$ is negative $(a = -1)$, the graph opens down so the second coordinate of the vertex, 2, is the maximum value of the function.

**c)** The range is $(-\infty, 2]$.

**d)** Since the graph opens down, function values increase as we approach the vertex from the left and decrease as we move to the right from the vertex. Thus the function is increasing on the interval $(-\infty, 7)$ and decreasing on $(7, \infty)$.

We can use a graphing calculator to do Example 4. Once we have graphed $y = -x^2 + 14x - 47$, we see that the graph opens down and thus has a maximum value. We can use the MAXIMUM feature to find the coordinates of the vertex. Using these coordinates, we can then find the maximum value and the range of the function along with the intervals on which the function is increasing or decreasing.

$y = -x^2 + 14x - 47$




## Applications

Many real-world situations involve finding the maximum or minimum value of a quadratic function.

Animation

You Try It

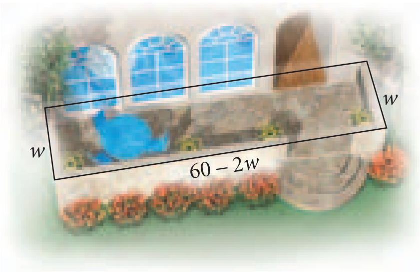
**EXAMPLE 5** *Maximizing Area.* A stonemason has enough stones to enclose a rectangular patio with 60 ft of stone wall. If the house forms one side of the rectangle, what is the maximum area that the mason can enclose? What should the dimensions of the patio be in order to yield this area?

PROBLEM-SOLVING STRATEGY

REVIEW SECTION 2.1.

*Solution* We will use the five-step problem-solving strategy.

1. **Familiarize.** We make a drawing of the situation, using $w$ to represent the width of the patio, in feet. Then $(60 - 2w)$ feet of stone is available for the length. Suppose the patio were 10 ft wide. It would then be $60 - 2 \cdot 10 = 40$ ft long. The area would be $(10 \text{ ft})(40 \text{ ft}) = 400 \text{ ft}^2$. If the patio were 12 ft wide, it would be $60 - 2 \cdot 12 = 36$ ft




long. The area would be $(12 \text{ ft})(36 \text{ ft}) = 432 \text{ ft}^2$. If it were 16 ft wide, it would be $60 - 2 \cdot 16 = 28$ ft long and the area would be $(16 \text{ ft})(28 \text{ ft}) = 448 \text{ ft}^2$. There are more combinations of length and width than we could possibly try. Instead we will find a function that represents the area and then determine the maximum value of the function.

2. **Translate.** Since the area of a rectangle is given by length times width, we have

$$\begin{aligned} A(w) &= (60 - 2w)w \qquad \mathbf{A = lw;\ l = 60 - 2w} \\ &= -2w^2 + 60w, \end{aligned}$$

where $A(w)$ is the area of the patio, in square feet, as a function of the width, $w$.

3. **Carry out.** To solve this problem, we need to determine the maximum value of $A(w)$ and find the dimensions for which that maximum occurs. Since $A$ is a quadratic function and $w^2$ has a negative coefficient, we know that the function has a maximum value that occurs at the vertex of the graph of the function. The first coordinate of the vertex, $(w, A(w))$, is

$$w = -\frac{b}{2a} = -\frac{60}{2(-2)} = 15 \text{ ft}.$$

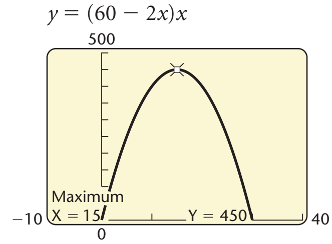
Thus, if $w = 15$ ft, then the length $l = 60 - 2 \cdot 15 = 30$ ft; and the area is $15 \cdot 30$, or $450 \text{ ft}^2$.

4. **Check.** As a partial check, we note that $450 \text{ ft}^2 > 448 \text{ ft}^2$, which is the largest area we found in a guess in the *Familiarize* step. As a more complete check, assuming that the function $A(w)$ is correct, we could examine a table of values for $A(w) = (60 - 2w)w$ and/or examine its graph.

| X | Y1 |
|---|---|
| 14.7 | 449.82 |
| 14.8 | 449.92 |
| 14.9 | 449.98 |
| 15 | 450 |
| 15.1 | 449.98 |
| 15.2 | 449.92 |
| 15.3 | 449.82 |

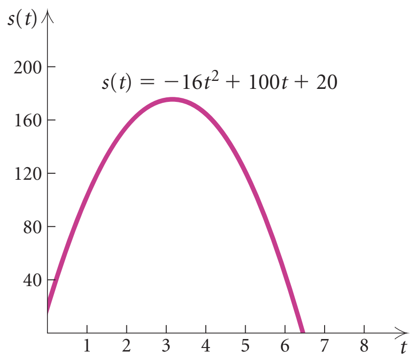
X = 15

$y = (60 - 2x)x$




5. **State.** The maximum possible area is $450 \text{ ft}^2$ when the patio is 15 ft wide and 30 ft long.

**EXAMPLE 6** *Height of a Rocket.* A model rocket is launched with an initial velocity of 100 ft/sec from the top of a hill that is 20 ft high. Its height $t$ seconds after it has been launched is given by the function $s(t) = -16t^2 + 100t + 20$. Determine the time at which the rocket reaches its maximum height and find the maximum height.




### Solution

**1., 2. Familiarize** and **Translate.** We are given the function in the statement of the problem.

**3. Carry out.** We need to find the maximum value of the function and the value of $t$ for which it occurs. Since $s(t)$ is a quadratic function and $t^2$ has a negative coefficient, we know that the maximum value of the function occurs at the vertex of the graph of the function. The first coordinate of the vertex gives the time $t$ at which the rocket reaches its maximum height. It is

$$t = -\frac{b}{2a} = -\frac{100}{2(-16)} = 3.125.$$

The second coordinate of the vertex gives the maximum height of the rocket. We substitute in the function to find it:

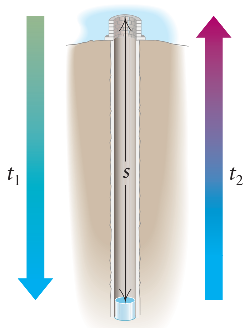
$$s(3.125) = -16(3.125)^2 + 100(3.125) + 20 = 176.25.$$

**4. Check.** As a check, we can complete the square to write the function in the form $s(t) = a(t - h)^2 + k$ and determine the coordinates of the vertex from this form of the function. We get

$$s(t) = -16(t - 3.125)^2 + 176.25.$$

This confirms that the vertex is $(3.125, 176.25)$, so the answer checks.

**5. State.** The rocket reaches a maximum height of 176.25 ft 3.125 sec after it has been launched.




**EXAMPLE 7** *Finding the Depth of a Well.* Two seconds after a chlorine tablet has been dropped into a well, a splash is heard. The speed of sound is 1100 ft/sec. How far is the top of the well from the water?

### Solution

**1. Familiarize.** We first make a drawing and label it with known and unknown information. We let $s$ = the depth of the well, in feet, $t_1$ = the time, in seconds, that it takes for the tablet to hit the water, and $t_2$ = the time, in seconds, that it takes for the sound to reach the top of the well. This gives us the equation

$$t_1 + t_2 = 2. \qquad (1)$$

**2. Translate.** Can we find any relationship between the two times and the distance $s$? Often in problem solving you may need to look up related formulas in a physics book, another mathematics book, or on the Internet. We find that the formula

$$s = 16t^2$$

gives the distance, in feet, that a dropped object falls in $t$ seconds. The time $t_1$ that it takes the tablet to hit the water can be found as follows:

$$s = 16t_1^2, \quad \text{or} \quad \frac{s}{16} = t_1^2, \text{ so } t_1 = \frac{\sqrt{s}}{4}. \qquad \text{Taking the positive square root} \qquad (2)$$

To find an expression for $t_2$, the time it takes the sound to travel to the top of the well, recall that *Distance* = *Rate* · *Time*. Thus,

$$s = 1100t_2, \quad \text{or} \quad t_2 = \frac{s}{1100}. \tag{3}$$

We now have expressions for $t_1$ and $t_2$, both in terms of $s$. Substituting into equation (1), we obtain

$$t_1 + t_2 = 2, \quad \text{or} \quad \frac{\sqrt{s}}{4} + \frac{s}{1100} = 2. \tag{4}$$

3. **Carry out.**

## Algebraic Solution

We solve equation (4) for $s$. Multiplying by 1100, we get

$$275\sqrt{s} + s = 2200, \quad \text{or} \quad s + 275\sqrt{s} - 2200 = 0.$$

This equation is reducible to quadratic with $u = \sqrt{s}$. Substituting, we get

$$u^2 + 275u - 2200 = 0.$$

Using the quadratic formula, we can solve for $u$:

$$\begin{aligned} u &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-275 + \sqrt{275^2 - 4 \cdot 1 \cdot (-2200)}}{2 \cdot 1} \\ &= \frac{-275 + \sqrt{84{,}425}}{2} \\ &\approx 7.78. \end{aligned}$$

**We want only the positive solution.**

Since $u \approx 7.78$, we have

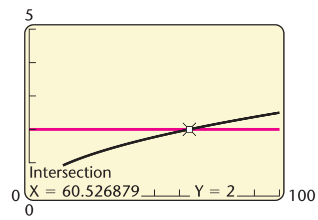
$$\sqrt{s} = 7.78$$

$$s \approx 60.5. \qquad \textbf{Squaring both sides}$$

## Graphical Solution

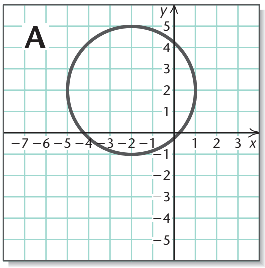
We use the Intersect method. It will probably require some trial and error to determine an appropriate window.

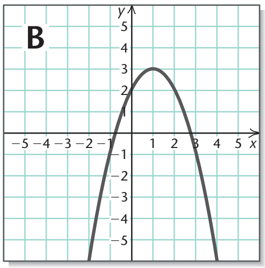
$$y_1 = \frac{\sqrt{x}}{4} + \frac{x}{1100}, \quad y_2 = 2$$

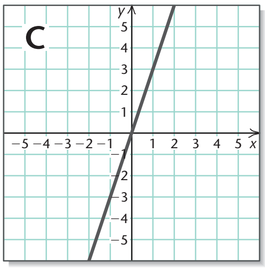


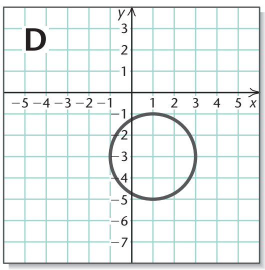

4. **Check.** To check, we can substitute 60.5 for $s$ in equation (4) and see that $t_1 + t_2 \approx 2$. We leave the mathematics for the student.

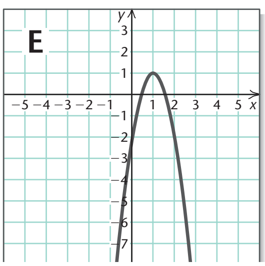
5. **State.** The top of the well is about 60.5 ft above the water.




# Visualizing the Graph

Match the equation with its graph.

1. $y = 3x$

2. $y = -(x - 1)^2 + 3$

3. $x^2 + 4x + y^2 - 4y - 1 = 0$

4. $y = 3$

5. $2x - 3y = 6$

6. $(x - 1)^2 + (y + 3)^2 = 4$

7. $y = -2x + 1$

8. $y = 2x^2 - x - 4$

9. $x = -2$

10. $y = -3x^2 + 6x - 2$

*Answers on page A-16*